Problem T–1

Let $\mathbb{Z}$ denote the set of all integers and $\mathbb{Z}_{>0}$ denote the set of all positive integers.

(a) A function $f\colon \mathbb{Z} \to \mathbb{Z}$ is called $\mathbb{Z}$-*good* if it satisfies $f(a^2+b) = f(b^2+a)$ for all $a, b \in \mathbb{Z}$. Determine the largest possible number of distinct values that can occur among $f(1), f(2), \ldots, f(2023)$, where f is a $\mathbb{Z}$-good function.

(b) A function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called $\mathbb{Z}_{>0}$-*good* if it satisfies $f(a^2+b) = f(b^2+a)$ for all $a, b \in \mathbb{Z}_{>0}$. Determine the largest possible number of distinct values that can occur among $f(1), f(2), \ldots, f(2023)$, where f is a $\mathbb{Z}_{>0}$-good function.

Problem T–2

Let a, b, c and d be positive real numbers with $abcd = 1$. Prove that

$$\frac{ab+1}{a+1} + \frac{bc+1}{b+1} + \frac{cd+1}{c+1} + \frac{da+1}{d+1} \geq 4,$$

and determine all quadruples (a, b, c, d) for which equality holds.

Problem T–3

Find the smallest integer b with the following property: For each way of colouring exactly b squares of an 8×8 chessboard green, one can place 7 bishops on 7 green squares so that no two bishops attack each other.

Remark. Two bishops attack each other if they are on the same diagonal.

Problem T–4

Let $c \geq 4$ be an even integer. In some football league, each team has a home uniform and an away uniform. Every home uniform is coloured in two different colours, and every away uniform is coloured in one colour. A team's away uniform cannot be coloured in one of the colours from the home uniform. There are at most c distinct colours on all of the uniforms. If two teams have the same two colours on their home uniforms, then they have different colours on their away uniforms.

We say a pair of uniforms is *clashing* if some colour appears on both of them. Suppose that for every team X in the league, there is no team Y in the league such that the home uniform of X is clashing with both uniforms of Y. Determine the maximum possible number of teams in the league.

Time: 5 hours
Time for questions: 60 min
Each problem is worth 8 points.
The order of the problems does not depend on their difficulty.

Problem T–5

We are given a convex quadrilateral $ABCD$ whose angles are not right. Assume there are points P, Q, R, S on its sides AB, BC, CD, DA, respectively, such that $PS \parallel BD$, $SQ \perp BC$, $PR \perp CD$. Furthermore, assume that the lines PR, SQ, and AC are concurrent. Prove that the points P, Q, R, S are concyclic.

Problem T–6

Let ABC be an acute triangle with $AB < AC$. Let J be the center of the A-excircle of ABC. Let D be the projection of J on line BC. The internal bisectors of angles BDJ and JDC intersect lines BJ and JC at X and Y, respectively. Segments XY and JD intersect at P. Let Q be the projection of A on line BC. Prove that the internal angle bisector of QAP is perpendicular to line XY.

Remark. The A-excircle of the triangle ABC is the circle outside the triangle which is tangent to the lines AB, AC, and the line segment BC.

Problem T–7

Find all positive integers n for which there exist positive integers $a > b$ satisfying

$$n = \frac{4ab}{a-b}.$$

Problem T–8

Let A and B be positive integers. Consider a sequence of positive integers $(x_n)_{n\geq 1}$ such that

$$x_{n+1} = A \cdot \gcd(x_n, x_{n-1}) + B \quad \text{for every } n \geq 2.$$

Prove that the sequence attains only finitely many different values.

Remark. We denote by $\gcd(a, b)$ the greatest common divisor of positive integers a and b.