EM 02

Solution Booklet

The Problem Selection Committee

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gratefully received

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Contents

I-1

Let $\mathbb{R}$ denote the set of all real numbers. For each pair (α, β) of nonnegative real numbers subject to $\alpha + \beta \geq 2$, determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x)f(y) \leq f(xy) + \alpha x + \beta y$$

for all real numbers x and y.

Solution. We know $f(x)f(y) \leq f(xy)+\alpha x+\beta y$ and by exchanging x and y we get $f(x)f(y) \leq f(xy) + \beta x + \alpha y$. Combining the two we get

$$f(x)f(y) \leq f(xy) + \gamma x + \gamma y, \tag{1}$$

where $\gamma = \frac{\alpha+\beta}{2}$. Notice that $\gamma \geq 1$.

Setting $x = y = -1$ in (1) we get $f(-1)^2 \leq f(1) - 2\gamma$, so $f(1) \geq 2\gamma$. Setting $x = y = 1$ in (1) we get $f(1)^2 \leq f(1) + 2\gamma$, so $f(1)^2 - f(1) \leq 2\gamma$. Since $f(1) \geq 2\gamma \geq 2$ and $t^2 - t$ is an increasing function for $t \geq 2$, we have $(2\gamma)^2 - 2\gamma \leq f(1)^2 - f(1) \leq 2\gamma$, hence $4\gamma^2 \leq 4\gamma$, so $\gamma \leq 1$. Therefore, $\gamma = 1$.

We know that $f(1) \geq 2$ and $f(1)^2 - f(1) \leq 2$, thus necessarily $f(1) = 2$. We also know $f(-1) \leq f(1) - 2\gamma = 0$, so $f(-1) = 0$.

Setting $x = z$, $y = 1$ in (1) we get $2f(z) \leq f(z) + z + 1$, so $f(z) \leq z + 1$. Setting $x = -z$, $y = -1$ in (1) we get $0 \leq f(z) - z - 1$, so $f(z) \geq z + 1$. It follows that the only function which can possibly satisfy the problem statement is

$$f(z) = z + 1. \tag{2}$$

It remains to check for which α and β this is indeed a solution.

Substituting f into original inequality, we get $(x + 1)(y + 1) \leq (xy + 1) + \alpha x + \beta y$, thus $(1 - \alpha)x + (1 - \beta)y \leq 0$. This holds for all x, y iff $\alpha = \beta = 1$. Hence, for $(\alpha, \beta) = (1, 1)$ the only solution is $f(z) = z + 1$ and for $(\alpha, \beta) \neq (1, 1)$ there are no solutions.

I-2

Find all integers $n \geq 3$ for which it is possible to draw n chords of one circle such that their $2n$ endpoints are pairwise distinct and each chord intersects precisely k other chords for:

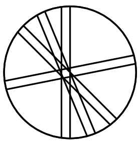
(a) $k = n - 2$,

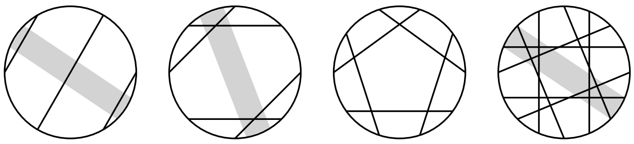
(b) $k = n - 3$.

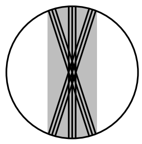
Remark. A chord of a circle is a line segment whose both endpoints lie on the circle.

Answer. (a) All even n. (b) All $n \geq 3$.

Solution. (a) Every chord avoids precisely one other chord, hence the avoiding chords form pairs and n must be even. On the other hand, for any even $n \geq 2$ the construction is simple (see the left figure).



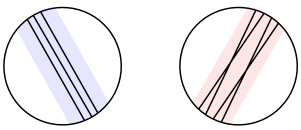




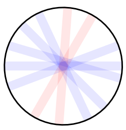
(b) For $n = 3, 4, 5, 8$ we can draw the chords as in the middle figure. From an admissible drawing with $n = 3, 4, 8$, we can build an admissible drawing with $n + 3k$ by adding k triples of parallel lines within the gray strip: Each existing chord crosses all the newly added chords, so it avoids precisely the 2 other chords it avoided before. Each newly added chord crosses all other chords except the other two chords in its triple.

Another Solution to (b). Another way to look at the construction in part (2) is as follows: Consider two "blocks" T and F of three and four chords, respectively, shown below in the left figure.

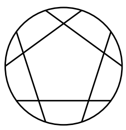
T F



$n = 5 \cdot 3 + 2 \cdot 4$



$n = 5$

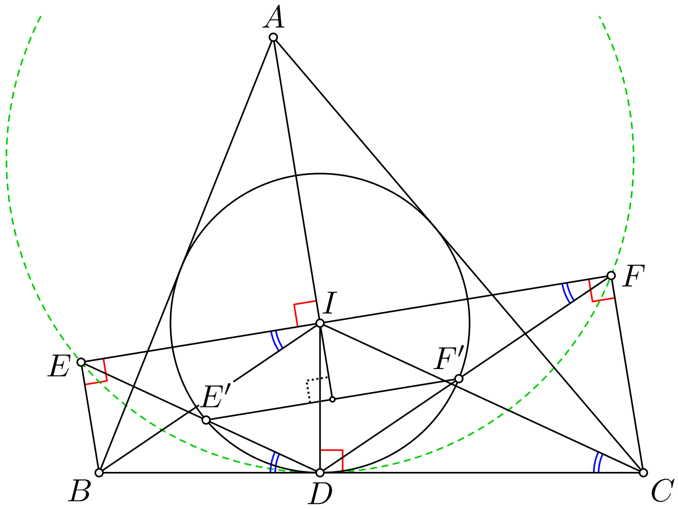


Note that each chord misses exactly two other chords from its block, so any time we place several blocks such that any two chords from different blocks cross, we obtain an admissible drawing (see the middle figure). Since all integers $n \geq 3$ except $n = 5$ can be expressed as a sum of several 3's and 4's (by a casework mod 3, or by the Frobenius coin problem aka the ChickenMcNugget theorem), it remains to find an admissible drawing for $n = 5$. For that, see the right figure.

I-3

Let ABC be a triangle with incenter I. The incircle ω of ABC is tangent to the line BC at point D. Denote by E and F the points satisfying $AI \parallel BE \parallel CF$ and $\angle BEI = \angle CFI = 90°$. Lines DE and DF intersect ω again at points E' and F', respectively. Prove that $E'F' \perp AI$.

Solution 1. Our goal essentially is to prove that the circumcirle of DEF is tangent to the incircle – that would immediately mean $EF \parallel E'F'$, which together with $EF \perp AI$ gives the desired result. In order to prove that we just need to show $\angle BDE = \angle EFD$.



Notice that quadrilaterals $BDIE$, $DCFI$ are cyclic due to right angles BDI, BEI, IDC, IFC. It it well-known that the circumcenter of BIC lies on AI (in fact, it is the second intersection of AI with the circumcircle of ABC). This means that line EIF is tangent to the circumcircle of BIC. With all these facts, we just need to perform simple angle chasing:

$$\angle BDE = \angle BIE = \angle ICB = \angle ICD = \angle IFD = \angle EFD.$$

Solution 2. Denote by B' and C' the projections of I on AC and AB, respectively. The right angles gives us that C, F, B', D lie on a circle. Angle chasing using $AI \parallel CF$ then gives:

$$\angle F'IB' = 2\angle F'DB' = 2\angle FDB' = 2\angle FCB' = 2\angle IAC = \angle BAC$$

Furthermore, we angle chase that:

$$\angle FIB' = 90° - \angle AIB' = \angle IAC = \frac{\angle BAC}{2}$$

So we get that:

$$\angle F'IF = \angle F'IB' - \angle FIB' = \angle BAC - \frac{\angle BAC}{2} = \frac{\angle BAC}{2}$$

Analogously, $\angle E'IE = \frac{\angle BAC}{2}$, so the line EIF is an external angle bisector in the triangle $E'IF'$. Since $AI \perp EF$, the line AI is an internal angle bisector in the triangle $E'IF'$. But triangle $E'IF'$ is isosceles, hence the line AI is also an altitude, so $AI \perp E'F'$.

I-4

Let n and m be positive integers. We call a set S of positive integers (n, m)-*good* if it satisfies the following three conditions:

(i) We have $m \in S$.

(ii) For all $a \in S$, all of the positive divisors of a are elements of S too.

(iii) For all mutually different numbers $a, b \in S$, we have $a^n + b^n \in S$.

Determine all pairs (n, m) such that the set of all positive integers is the only (n, m)-good set.

Answer. The set $\mathbb{Z}_{\geq 1}$ is the only (m, n)-good set if and only if n is odd and $m \geq 2$.

Solution. For $m = 1$ we have that $\{1\}$ is (m, n)-good. For the rest of the solution we assume $m \geq 2$.

- n is odd
  Let S be (m, n)-good set. Since $x+y \mid x^n+y^n$, for $x, y \in S$ with $x \neq y$ we have $x+y \in S$. Since $1 \mid m$, it implies $1 \in S$ and also $m+1 \in S$. By induction, all positive integers greater than m are in S. Moreover, every postive integer smaller than m has a multiple which is greater than m. This implies that $S = \mathbb{Z}_{\geq 1}$.

- n is even
  Let $n = 2k$ and let p be a prime coprime to m such that $p \equiv 3 \mod 4$. Such prime exists since there are infinitely many primes with remainder 3 modulo 4 (this well-known fact follows for example from Dirichlet's theorem). Let $S = \{x \in \mathbb{Z}_{\geq 1} : p \nmid x\}$. We will show that S is (m, n)-good set. Clearly, the first two conditions are satisfied. Consider two distinct elements $x, y \in S$ such that

  $$x^{2k} \equiv -y^{2k} \mod p.$$

  By exponentiating this congruence to the power of $\frac{p-1}{2}$, we obtain

  $$x^{k(p-1)} \equiv (-1)^{\frac{p-1}{2}} \cdot y^{k(p-1)} \mod p,$$

  form which it follows that

  $$1 \equiv (-1)^{\frac{p-1}{2}} \mod p.$$

  Since $p \equiv 3 \mod 4$, $(p-1)/2$ is an odd number, thus $(-1)^{\frac{p-1}{2}} = -1$ is a contradiction. Therefore, S is (m, n)-good set.

Comment. The end of the solution can be replaced by stating the well-known fact that for $p \equiv 3 \pmod 4$ we have $p \mid x^2 + y^2 \Rightarrow p \mid x, y$.

T-1

Let $\mathbb{Z}$ denote the set of all integers and $\mathbb{Z}_{>0}$ denote the set of all positive integers.

(a) A function $f\colon \mathbb{Z} \to \mathbb{Z}$ is called $\mathbb{Z}$-*good* if it satisfies $f(a^2+b) = f(b^2+a)$ for all $a, b \in \mathbb{Z}$. Determine the largest possible number of distinct values that can occur among $f(1), f(2), \ldots, f(2023)$, where f is a $\mathbb{Z}$-good function.

(b) A function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called $\mathbb{Z}_{>0}$-*good* if it satisfies $f(a^2+b) = f(b^2+a)$ for all $a, b \in \mathbb{Z}_{>0}$. Determine the largest possible number of distinct values that can occur among $f(1), f(2), \ldots, f(2023)$, where f is a $\mathbb{Z}_{>0}$-good function.

Solution. The answer is (a) 2 and (b) 1077.

(a) Note that

$$f(a^2+b) = f(b^2+a) = f((-b)^2+a) = f(a^2-b).$$

In particular, by setting $a \in \{0, 1\}$ we get $f(b) = f(-b)$ and $f(1+b) = f(1-b)$. This then yields

$$f(2+b) = f(1+(1+b)) = f(1-(1+b)) = f(-b) = f(b),$$

hence by induction the function must be constant on even integers and (separately) on odd integers. On the other hand, a function $f(n) = n \pmod 2$ satisfies the requirements and attains 2 distinct values on $\{1, \ldots, 2023\}$.

(b) Given two positive integers $m < n$, we say that m is a *parent* of n if there exist two positive integers a, b such that $a^2+b = m$ and $b^2+a = n$. Note that:

(a) If $m = a^2+b$ and $n = b^2+a$ (for positive integers a, b, m, n) then $m < n$ if and only if $a < b$: This follows from the fact that $x^2 - x$ is increasing on $\mathbb{N}$.

(b) The numbers in $S = \{1, 2, 3, 4\}$ do not have a parent: This is a simple check.

(c) For any $n \geq 2$, the $n-1$ numbers in $A_n = \{n^2+1, \ldots, n^2+(n-1)\}$ each have a unique parent: Let $1 \leq k \leq n-1$ and assume that $n^2+k = a^2+b$ with $a > b > 0$. Then either $n = a$, $k = b$ (in which case we obtain a parent $k^2+n < n^2+k$) or $a < n$, in which case we get $b \geq n^2+k-(n-1)^2 \geq 2n-1 > a$, a contradiction.

(d) For any $n \geq 2$, the $n+2$ numbers in $B_n = \{n^2+n, \ldots, n^2+2n+1\}$ do not have a parent: Let $n \leq k \leq 2n+1$ and assume that $n^2+k = a^2+b$ with $a > b > 0$. Since $n^2+k < (n+1)^2+1$, we must have $a \leq n$ but then $b \geq k \geq n \geq a$, a contradiction.

Note that as $n \in \mathbb{N}$ varies, the sets A_n, B_n form a partition of $\mathbb{N} \setminus S$, hence each positive integer has at most one parent. In other words, if we process the positive integers in increasing order, for any currently processed integer n there will always be at most one parent, and thus at most one requirement on which value to assign to $f(n)$. Therefore, as we process the integers, to any integer n without a parent we can always safely assign any value $f(n)$. (Clearly, when n has a parent m, we must assign $f(n) = f(m)$.) The answer is the number of integers $n \in \{1, \ldots, 2023\}$ with no parent. Since $2023 = 44^2 + 87$, this is

$$|S| + \left(\sum_{i=2}^{43} (i+2) \right) + (87 - 44 + 1) = 1077.$$

Comment. It is possible to ask only for the answer to the part (b).

T-2

Let a, b, c and d be positive real numbers with $abcd = 1$. Prove that

$$\frac{ab+1}{a+1} + \frac{bc+1}{b+1} + \frac{cd+1}{c+1} + \frac{da+1}{d+1} \geq 4,$$

and determine all quadruples (a, b, c, d) for which equality holds.

Solution. By assumption we have

$$\begin{aligned}\frac{ab+1}{a+1} + \frac{bc+1}{b+1} + \frac{cd+1}{c+1} + \frac{da+1}{d+1} &= \left(\frac{ab+1}{a+1} + \frac{cd+abcd}{c+1}\right) + \left(\frac{bc+1}{b+1} + \frac{da+abcd}{d+1}\right) = \\ &= (ab+1)\left(\frac{1}{a+1} + \frac{cd}{c+1}\right) + (bc+1)\left(\frac{1}{b+1} + \frac{ad}{d+1}\right) = \\ &= (ab+1)\left(\frac{1}{a+1} + \frac{1}{abc+ab}\right) + (bc+1)\left(\frac{1}{b+1} + \frac{1}{bcd+bc}\right).\end{aligned}$$

When estimating the parentheses by the inequality of the arithmetic-harmonic mean (i. e. $1/u + 1/v \geq 4/(u+v)$ for positive real numbers u and v with equality iff $u = v$), we see that the above expression is at least

$$\begin{aligned}(ab+1)\frac{4}{a+1+abc+ab} + (bc+1)\frac{4}{b+1+bcd+bc} &= \frac{4(ab+1)}{a+1+abc+ab} + \frac{4(bc+1)\cdot a}{(b+1+bcd+bc)\cdot a} = \\ &= \frac{4(ab+1)}{a+1+abc+ab} + \frac{4(abc+a)}{ab+a+1+abc} = \\ &= \frac{4(ab+1+abc+a)}{1+a+ab+abc} = 4.\end{aligned}$$

Equality holds iff $a + 1 = abc + ab$ and $b + 1 = bcd + bc \Leftrightarrow ab + a = 1 + abc$. Addition of these equations yields $2a = 2abc$, that is $bc = 1$. Plugging this into the first equation, we also get $1 = ab$. Therefore $ab = bc = cd = 1$, which implies $a = c$ and $b = d = 1/a$. Indeed, these conditions suffice, as is readily checked. Thus, equality holds iff $(a, b, c, d) = (t, \frac{1}{t}, t, \frac{1}{t})$ for some positive real t.

T-3

Find the smallest integer b with the following property: For each way of colouring exactly b squares of an 8×8 chessboard green, one can place 7 bishops on 7 green squares so that no two bishops attack each other.

Remark. Two bishops attack each other if they are on the same diagonal.

Answer. 41

Solution. Let us place 40 bishops on 6 diagonals as shown in Figure 2. If we select any 7 of the placed bishops, by Pigeon hole principle, at least two of the selected bishops are on the same diagonal, so they attack each other. Thus, the number b of selected bishops is at least 41.

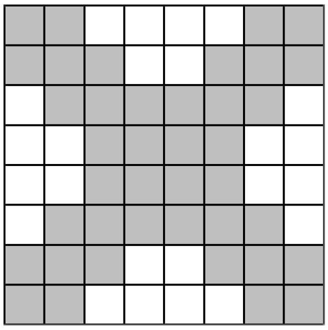


Figure 1

Now, suppose for a contrary, that there is a placement of 41 bishops such there are not 7 non-attacking bishops. Divide all tiles to 8 groups as shown in Figure **??**. It is easy to see that any two bishops belonging to the same group do not attack each other. Therefore, each group contains at most 6 bishops. Moreover, groups 7 and 8 contain at most 2 bishops due to their size. Therefore, we have at most $6 \cdot 6 + 2 \cdot 2 = 40$ bishops, which is a contradiction. Therefore, from any placement of 41 bishops, it is possible to select some 7 of them such that no two attack each other. This, together with the lower bound of $b \geq 41$ finishes this solution.

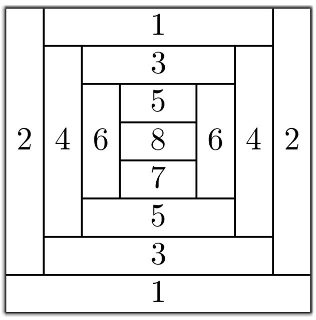


Figure 2

Comment (1). A weaker upper bound of 49 can be shown as follows: Consider a placement of 49. bishops. We have 8 rows and $49/8 > 6$, so by Pigeon hole principle, there is a row with at least 7 bishops. Clearly, bishops in the same row do not attack each other.

Comment (2). This problem can be generalised for larger dimensions of the chessboard and also larger number of sought non-attacking bishops.

T-4

Let $c \geq 4$ be an even integer. In some football league, each team has a home uniform and an away uniform. Every home uniform is coloured in two different colours, and every away uniform is coloured in one colour. A team's away uniform cannot be coloured in one of the colours from the home uniform. There are at most c distinct colours on all of the uniforms. If two teams have the same two colours on their home uniforms, then they have different colours on their away uniforms.

We say a pair of uniforms is *clashing* if some colour appears on both of them. Suppose that for every team X in the league, there is no team Y in the league such that the home uniform of X is clashing with both uniforms of Y. Determine the maximum possible number of teams in the league.

Answer. We claim the answer is $\frac{n^3}{8} - \frac{n^2}{4}$.

Solution. We first give an example of a league with $\frac{n^3}{8} - \frac{n^2}{4}$ teams.

Split the colours in two sets of size $n/2$. Let $m = n/2$ and let $c_1, \ldots, c_m$ and $d_1, \ldots, d_m$ be the colours in those sets.

Consider all pairs of kits of the form $(\{c_i, c_j\}, d_k)$ or $(\{d_i, d_j\}, c_k)$, where $i < j$ and $1 \leq i, j, k \leq m$. There are $2 \cdot \binom{m}{2} \cdot m = m^3 - m^2 = \frac{n^3}{8} - \frac{n^2}{4}$ such pairs of kits. We claim that this construction is valid.

Consider any pair of kits $(\{c_i, c_j\}, d_k)$. Then for any other team of the form $(\{c_a, c_b\}, d_u)$, the kit d_u is not clashing with the home kit $\{c_i, c_j\}$. Furthermore, for any team of the form $(\{d_a, d_b\}, c_u)$ the kit $\{d_a, d_b\}$ is not clashing with the home kit $\{c_i, c_j\}$. Thus, the construction is valid.

We now prove that there is no larger league. Consider any colour c. Take any other colour d. If there is a team whose home kit is $\{c, d\}$, then there is no team whose home kit contains c and whose away kit is d. Conversely, if there is a team whose home kit contains c and whose away kit is d, then there is no team whose home kit is $\{c, d\}$.

Let $A(c)$ be the number of colours d such that there is a home kit of the form $\{c, d\}$, and let $B(c)$ the number of colours d such that there is a team whose home kit contains c and whose away kit is d.

From the observation we made, $A(c) + B(c) \leq n - 1$. The number of teams whose home kit contains the colour c is at most

$$A(c)B(c) \leq \frac{n-2}{2} \cdot \frac{n-1}{2} = \frac{n^2}{4} - \frac{n}{2},$$

where the inequality follows from the fact that the function $x \mapsto x(n-1-x)$ is increasing on $(0,(n-1)/2)$ and decreasing on $((n-1)/2, n-1)$.

Summing up over all colours c and dividing by 2 since we counted each home kit twice, we obtain that the number of teams is at most

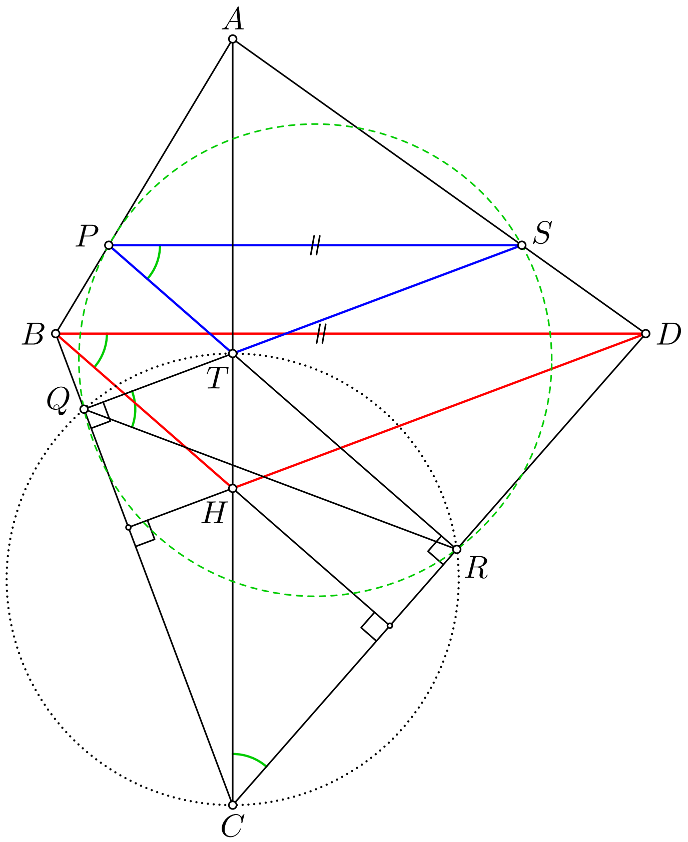
$$\frac{1}{2}\sum_{c} A(c)B(c) \leq \frac{n}{2} \cdot \left(\frac{n^2}{4} - \frac{n}{2}\right) = \frac{n^3}{8} - \frac{n^2}{4}.$$

T-5

We are given a convex quadrilateral $ABCD$ whose angles are not right. Assume there are points P, Q, R, S on its sides AB, BC, CD, DA, respectively, such that $PS \parallel BD$, $SQ \perp BC$, $PR \perp CD$. Furthermore, assume that the lines PR, SQ, and AC are concurrent. Prove that the points P, Q, R, S are concyclic.

Solution. Let the interesection point of PR, QS, AC be T and let H be the orthocenter of BCD. Since $\angle BCD$ is not right, $H \neq C$. Notice that triangles HBD and TPS are homothetic due to their corresponding sides being parallel. This means that HT, BP, DS are concurrent. Since $H \neq C$, this gives that H lies on line ATC.

Due to parallel lines, we have $\angle RPS = \angle TPS = \angle HBD$. Since H is the orthocenter, we have $\angle HBD = \angle DCH = \angle RCT$. Finally, note that quadrilateral $QTRC$ is cyclic due to right angles at Q and R, which gives $\angle RCT = \angle RQT = \angle RQS$. Together, we have $\angle RPS = \angle RQS$, which finishes the proof.



T-6

Let ABC be an acute triangle with $AB < AC$. Let J be the center of the A-excircle of ABC. Let D be the projection of J on line BC. The internal bisectors of angles BDJ and JDC intersect lines BJ and JC at X and Y, respectively. Segments XY and JD intersect at P. Let Q be the projection of A on line BC. Prove that the internal angle bisector of QAP is perpendicular to line XY.

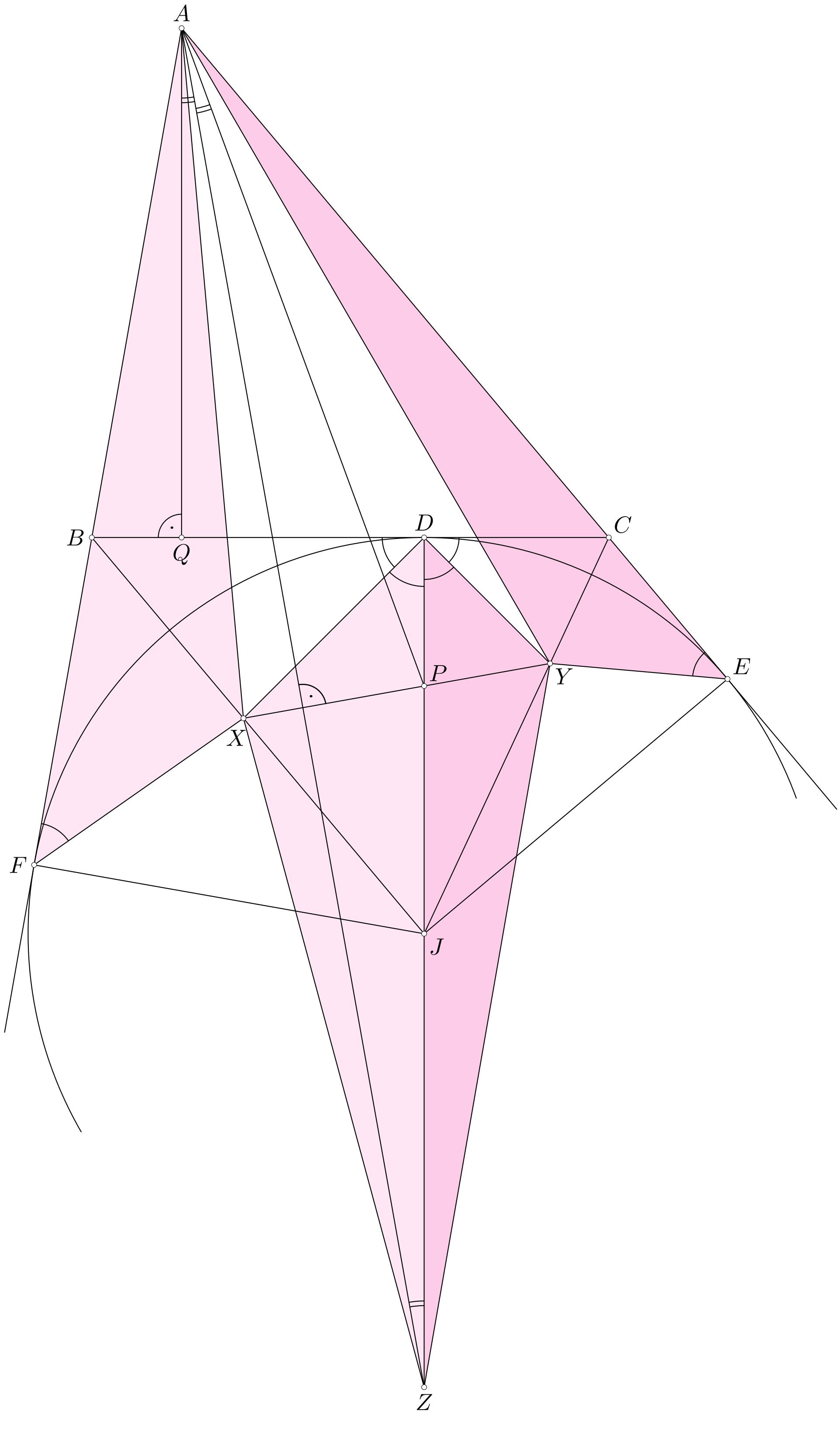
Remark. The A-excircle of the triangle ABC is the circle outside the triangle which is tangent to the lines AB, AC, and the line segment BC.

Solution. Let E and F be the points symmetric to D with respect to lines JC and BJ, respectively. Then D, E and F are the points of tangency of the A-excircle of ABC with BC, CA and AB, respectively. In particular, $AE = AF$.

Let Z be the point on ray $DP^{\rightarrow}$ such that $DZ = AE$. Note that $CE = CD$ and $\angle CEY = \angle YDC = \angle ZDY$. It follows by SAS that $\triangle AEY \equiv \triangle ZDY$. Therefore $AY = YZ$.

Similarly, $DZ = AF$, $FX = DX$, and $\angle XFA = \angle BDX = \angle XDZ$. Thus by SAS $\triangle XFA \equiv \triangle XDZ$. Hence $AX = XZ$.

Since $AX = XZ$ and $AY = YZ$, it follows that XY is the perpendicular bisector of AZ. In particular, $AP = PZ$. Hence $\angle ZAP = \angle PZA = \angle QAZ$, where the latter equality holds true as $AQ \parallel DZ$. This shows that AZ is the internal angle bisector of $\angle QAP$ and we are done because $XY \perp AZ$.


A
B
Q
D
C
E
P
Y
X
F
J
Z

T-7

Find all positive integers n for which there exist positive integers $a > b$ satisfying

$$n = \frac{4ab}{a-b}.$$

Answer. Any composite $n \neq 4$.

Solution. We say that n is *good* if there exist such positive integers a and b (and *bad* otherwise).

First, we show that $n = 4$ is bad. Suppose otherwise. Then there exist positive integers a, b such that

$$4 = \frac{4ab}{a-b}.$$

But this rewrites as $(a+1)(b-1) = -1$ and the left-hand side is non-negative for any two positive integers a, b, a contradiction.

Now consider $n \geq 6$. We make three observations.

First, note that if n is good then any its multiple $n' = k \cdot n$ is also good – it suffices to take $a' = k \cdot a$ and $b' = k \cdot b$. In particular, for $a = 2$, $b = 1$ we have $4ab/(a-b) = 8$, hence all multiples of 8 are good.

Second, take any positive integer t of the form $t = 4k+1$. By setting $a = t$, $b = 1$ we get

$$\frac{4ab}{a-b} = \frac{4 \cdot t}{4k} = \frac{t}{k},$$

hence by setting $a = t \cdot k$, $b = k$ we obtain that any multiple of any number of the form $t = 4k+1$ is good.

Third, likewise, take any positive integer t of the form $t = 4k-1$. By setting $a = t$, $b = 1$ we get

$$\frac{4ab}{a-b} = \frac{4 \cdot t}{4k-2} = \frac{2t}{2k-1},$$

hence every multiple of *double* of any number of the form $t = 4k-1$ is good.

Now we combine the observations. Consider any composite number $n \geq 6$. If the prime factorization of n contains a prime of the form $4k+1$ then n is good (by the second observation). Similarly, if n contains at least two (not necessarily distinct) primes of the form $4k-1$ then their product is of the form $4k+1$, and thus n is good (by the second observation). If n contains a prime of the form $4k-1$ and it is even, then n is also good (by the third observation). Thus, it remains to consider the powers of 2. But we know that all multiples of 8 are good (by the first observation) and 4 is bad, so we are done.

T-8

Let A and B be positive integers. Consider a sequence of positive integers $(x_n)_{n\geq 1}$ such that

$$x_{n+1} = A \cdot \gcd(x_n, x_{n-1}) + B \quad \text{for every } n \geq 2.$$

Prove that the sequence attains only finitely many different values.

Remark. We denote by $\gcd(a,b)$ the greatest common divisor of positive integers a and b.

Solution. Let $n \geq 2$ be a positive integer such that $x_{n+1} > x_n$. Then

$$\frac{x_n}{\gcd(x_n, x_{n-1})} < \frac{x_{n+1}}{\gcd(x_n, x_{n-1})} = A + \frac{B}{\gcd(x_n, x_{n-1})} \leq A + B.$$

Furthermore,

$$\begin{aligned}
\gcd(x_n, x_{n+1}) &= \gcd(x_n, A\gcd(x_n, x_{n-1}) + B) \\
&\leq \gcd(\gcd(x_n, x_{n-1}), A\gcd(x_n, x_{n-1}) + B) \cdot \gcd\left(\frac{x_n}{\gcd(x_n, x_{n-1})}, A\gcd(x_n, x_{n-1}) + B\right) \\
&= \gcd(\gcd(x_n, x_{n-1}), B) \cdot \gcd\left(\frac{x_n}{\gcd(x_n, x_{n-1})}, A\gcd(x_n, x_{n-1}) + B\right),
\end{aligned}$$

where we first used the fact that $\gcd(ab, c) \leq \gcd(a, c)\gcd(b, c)$, and then used the Euclidean algorithm on the left factor of the right hand side.

Now we'll bound each of the factors by constants. The left factor is not greater than B, and the right factor is not greater than $\frac{x_n}{\gcd(x_n, x_{n-1})}$, which is less than or equal to $A + B$. We conclude that

$$\gcd(x_{n+1}, x_n) \leq B(A + B),$$

which implies $x_{n+2} \leq AB(A + B) + B$.

We say an element of the sequence is big if it is greater than $AB(A + B) + B$, and small otherwise. The sequence is either eventually decreasing, or it contains a small element, since x_{n+2} is small whenever $x_{n+1} > x_n$.

We've proven that whenever x_j is small, either $x_{j+1} \leq x_j$ and hence x_{j+1} is also small, or x_{j+2} is small. This means that from some point in the sequence onwards there are no two consecutive big elements, but then all but finitely many elements of the sequence are not greater than $A(AB(A + B) + B) + B$, because either they're small or the previous element of the sequence is small.

Since we bounded all but finitely many elements of the sequence, the claim is proven.